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# On Mutually Nearest and Mutually Furthest Points of Sets in Banach Spaces

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Let $A$ be a nonempty closed bounded subset of a uniformly convex Banach space $\mathbb{E}$. Let $\mathscr{C}(\mathbb{E})$ denote the space of all nonempty closed convex and bounded subsets of $\mathbb{E}$, endowed with the Hausdorff metric. We prove that the set of all $X \in \mathscr{C}(\mathbb{E})$ such that the maximization problem $\max(A, X)$ is well posed is a $G_\delta$ dense subset of $\mathscr{C}(\mathbb{E})$. A similar result is proved for the minimization problem $\min(A, X)$, with $X$ in an appropriate subspace of $\mathscr{C}(\mathbb{E})$. © 1992 Academic Press, Inc.

## 1. Introduction and Preliminaries

Let $\mathbb{E}$ be a real Banach space. We denote by $\mathscr{B}(\mathbb{E})$ the space of all nonempty closed bounded subsets of $\mathbb{E}$. For $X$, $Y \in \mathscr{B}(\mathbb{E})$, we set

$$\lambda_{XY} = \inf\{\|x - y\| \mid x \in X,\ y \in Y\},$$
$$\mu_{XY} = \sup\{\|x - y\| \mid x \in X,\ y \in Y\}.$$

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Given $X$, $Y \in \mathscr{B}(\mathbb{E})$, we consider the *minimization* (resp. *maximization*) problem, denoted $\min(X, Y)$ (resp. $\max(X, Y)$), which consists in finding points $x_0 \in X$ and $y_0 \in Y$ such that $\|x_0 - y_0\| = \lambda_{XY}$ (resp. $\|x_0 - y_0\| = \mu_{XY}$). Any such pair $(x_0, y_0)$ is called a *solution* of the corresponding problem. Moreover, any sequence $\{(x_n, y_n)\}$, $x_n \in X$, $y_n \in Y$, such that $\lim_{n \to \infty} \|x_n - y_n\| = \lambda_{XY}$ (resp. $\lim_{n \to \infty} \|x_n - y_n\| = \mu_{XY}$) is called a *minimizing* (resp. *maximizing*) sequence. A minimization (resp. maximization) problem is said to be *well posed* if it has a unique solution $(x_0, y_0)$, and every minimizing (resp. maximizing) sequence converges to $(x_0, y_0)$.

Let $M$ be a metric space with distance $d$. For any $u \in M$ and $r > 0$ we set $B_M(u, r) = \{x \in M \mid d(x, u) < r\}$ and $\tilde{B}_M(u, r) = \{x \in M \mid d(x, u) \leqslant r\}$. If $X \subset M$, by $\bar{X}$ and $\operatorname{diam} X$ $(X \neq \phi)$ we mean the closure of $X$ and the diameter of $X$, respectively. As usual, if $X \subset \mathbb{E}$, $\overline{\operatorname{co}}\, X$ stands for the closed convex hull of $X$. We put, for short, $B = B_{\mathbb{E}}(0, 1)$ and $\tilde{B} = \tilde{B}_{\mathbb{E}}(0, 1)$.

We set

$$\mathscr{C}(\mathbb{E}) = \{X \subset \mathbb{E} \mid X \text{ is nonempty, convex, closed, bounded}\}.$$

In the sequel, we suppose the space $\mathscr{C}(\mathbb{E})$ to be endowed with the Hausdorff distance $h$. As is well known, under such metric, $\mathscr{C}(\mathbb{E})$ is complete.

In this note we consider problems of minimization, $\min(A, X)$, and of maximization, $\max(A, X)$, where $A \in \mathscr{B}(\mathbb{E})$, $X \in \mathscr{C}(\mathbb{E})$, and $\mathbb{E}$ is uniformly convex. More precisely, for a fixed $A \in \mathscr{B}(\mathbb{E})$, set $\mathscr{C}_A(\mathbb{E}) = \overline{\{X \in \mathscr{C}(\mathbb{E}) \mid \lambda_{AX} > 0\}}$. Then, it is proved (Theorem 3.3) that the set of all $X \in \mathscr{C}_A(\mathbb{E})$, such that the minimization problem $\min(A, X)$ is well posed, is a dense $G_\delta$-subset of $\mathscr{C}_A(\mathbb{E})$. Furthermore, it is shown (Theorem 4.3) that the set of all $X \in \mathscr{C}(\mathbb{E})$, such that the maximization problem $\max(A, X)$ is well posed, is a dense $G_\delta$-subset of $\mathscr{C}(\mathbb{E})$.

The problems considered in this note are in the spirit of Stečkin [22]. Some further developments of Stečkin's ideas, also in other directions, can be found in [4–6, 12, 14–21] and in the monograph [10], by Dontchev and Zolezzi. Recently, a generic theorem on points of single valuedness of the proximity map for convex sets has been established by Beer and Pai [3], in a setting different from ours. Some other generic results in spaces of convex sets can be found in [2, 8].

## 2. Auxiliary Results

Let $X \in \mathscr{B}(\mathbb{E})$ and $z \in \mathbb{E}$ be arbitrary. We set

$$d(z, X) = \inf\{\|z - x\| \mid x \in X\},$$

$$e(z, X) = \sup\{\|z - x\| \mid x \in X\}.$$

For $X, Y \in \mathscr{B}(\mathbb{E})$ and $\sigma > 0$, we set

$$L_{X,Y}(\sigma) = \{x \in X \mid d(x, Y) \leqslant \lambda_{XY} + \sigma\},$$
$$M_{X,Y}(\sigma) = \{x \in X \mid e(x, Y) \geqslant \mu_{XY} - \sigma\}.$$

The sets $L_{XY}(\sigma)$, $M_{XY}(\sigma)$ are nonempty, closed, and satisfy $L_{XY}(\sigma) \subset L_{XY}(\sigma')$, $M_{XY}(\sigma) \subset M_{XY}(\sigma')$, if $0 < \sigma < \sigma'$.

PROPOSITION 2.1. *Let $X, Y \in \mathscr{B}(\mathbb{E})$ and $z \in \mathbb{E}$ be arbitrary. Then we have*

$$\lambda_{XY} \leqslant d(z, X) + d(z, Y), \tag{2.1}$$

$$\mu_{XY} \geqslant e(z, Y) - d(z, X). \tag{2.2}$$

*Proof.* Both inequalities follow easily from the definitions.

PROPOSITION 2.2. *Let $X, Y \in \mathscr{B}(\mathbb{E})$ be arbitrary. Then the problem* $\min(X, Y)$ (*resp.* $\max(X, Y)$) *is well posed if and only if*

$$\inf_{\sigma > 0} \operatorname{diam} L_{XY}(\sigma) = 0 \quad \textit{and} \quad \inf_{\sigma > 0} \operatorname{diam} L_{YX}(\sigma) = 0$$

$$(\textit{resp.} \inf_{\sigma > 0} \operatorname{diam} M_{XY}(\sigma) = 0 \quad \textit{and} \quad \inf_{\sigma > 0} \operatorname{diam} M_{YX}(\sigma) = 0).$$

*Proof.* This is an easy adaptation of an argument due to Furi and Vignoli [13].

The following proposition is a variant of a result due to Zabreĭko and Krasnosel'skiĭ [23] and Daneš [7] (see also [8]).

PROPOSITION 2.3. *Let $X \in \mathscr{C}(\mathbb{E})$, $\varepsilon > 0$, and $r > 0$ be arbitrary. Then there exists $0 < \tau_0 < r$ such that for every $u \in \mathbb{E}$, with $d(u, X) \geqslant r$, and for every $0 < \tau \leqslant \tau_0$, we have*

$$\operatorname{diam} C_{X,u}(\tau) < \varepsilon,$$

*where*

$$C_{X,u}(\tau) = [\overline{\mathrm{co}}(X \cup \{u\})] \backslash [X + (d(u, X) - \tau) B]. \tag{2.3}$$

PROPOSITION 2.4. *Let $\mathbb{E}$ be a uniformly convex Banach space. Let $\varepsilon > 0$ and let $r_0$, $r > 0$, with $r < r_0$, be arbitrary. Then there exists $0 < \sigma_0 < r$ such that for every $x, y \in \mathbb{E}$, with $\|y - x\| = r$, and for every $r < r' \leqslant r_0$ and $0 < \sigma \leqslant \sigma_0$, we have*

$$\operatorname{diam} D(x, y; r', \sigma) < \varepsilon,$$

*where*

$$D(x, y; r', \sigma) = \tilde{B}_{\mathbb{E}}(y, r' - \|y - x\| + \sigma) \backslash B_{\mathbb{E}}(x, r').$$

*Proof.* Let $\varepsilon > 0$ and $0 < r < r_0$ be given. Let $x, y \in \mathbb{E}$ satisfy $\|y - x\| = r$. Let $r < r' \leqslant r_0$ be arbitrary and let $y' = (x + y)/2$. We have $D(x, y; r', \sigma) \subset D(x, y'; r', \sigma)$, $\sigma > 0$. Moreover, by [9, Lemma 2.1], if $0 < \sigma \leqslant 2 \|y' - x\|$, we have

$$\operatorname{diam} D(x, y'; r', \sigma) \leqslant 2\sigma + 2(r' - \|y' - x\|)\, \delta^* \left( \frac{\sigma}{2 \|y' - x\|} \right)$$

$$\leqslant 2\sigma + (2r_0 - r)\, \delta^* \left( \frac{\sigma}{r} \right),$$

where, for $\eta > 0$, $\delta^*(\eta) = \sup\{\varepsilon \mid 0 < \varepsilon \leqslant 2 \text{ and } \delta(\varepsilon) \leqslant \eta\}$ and $\delta$ denotes the modulus of convexity of $\mathbb{E}$. Since the last term in the above inequality vanishes as $\sigma \to 0$, to complete the proof it suffices to choose $\sigma_0 > 0$ such that $2\sigma_0 + (2r_0 - r)\, \delta^*(\sigma_0/r) < \varepsilon$.

## 3. Minimization Problems

In this section $\mathbb{E}$ denotes a uniformly convex Banach space. Let $A$ be a fixed nonempty closed bounded subset of $\mathbb{E}$. We put, for short, $\lambda_X = \lambda_{AX}$, $X \in \mathscr{B}(\mathbb{E})$. Define

$$\mathscr{C}_A(\mathbb{E}) = \overline{\{X \in \mathscr{C}(\mathbb{E}) \mid \lambda_X > 0\}}.$$

Under the Hausdorff distance, $\mathscr{C}_A(\mathbb{E})$ is a complete metric space.

For each $k \in \mathbb{N}$ set $\varepsilon_k = 1/k$, and define

$$\mathscr{L}_k = \{X \in \mathscr{C}_A(\mathbb{E}) \mid \inf_{\sigma > 0} \operatorname{diam} L_{XA}(\sigma) < \varepsilon_k \text{ and } \inf_{\sigma > 0} \operatorname{diam} L_{AX}(\sigma) < \varepsilon_k\}.$$

To prove the main result of this section, Theorem 3.3, we state two lemmas, whose proofs will be given later.

Lemma 3.1. *$\mathscr{L}_k$ is dense in $\mathscr{C}_A(\mathbb{E})$.*

Lemma 3.2. *$\mathscr{L}_k$ is open in $\mathscr{C}_A(\mathbb{E})$.*

Theorem 3.3. *Let $\mathbb{E}$ be a uniformly convex Banach space. Let $A \in \mathscr{B}(\mathbb{E})$. Then the set*

$$\mathscr{V} = \{X \in \mathscr{C}_A(\mathbb{E}) \mid \min(A, X) \textit{ is well posed}\}$$

*is a dense $G_\delta$-subset of $\mathscr{C}_A(\mathbb{E})$.*

*Proof.* By Lemmas 3.1 and 3.2, the set

$$\mathscr{L}_0 = \bigcap_{k \in \mathbb{N}} \mathscr{L}_k$$

is a dense $G_\delta$-subset of $\mathscr{C}_A(\mathbb{E})$. Moreover, by Proposition 2.2, we have $\mathscr{V} = \mathscr{L}_0$. Hence $\mathscr{V}$ is a dense $G_\delta$-subset of $\mathscr{C}_A(\mathbb{E})$, completing the proof.

*Remark* 3.4. If $A = \tilde{B}$ and $X_0 = \frac{1}{2}\tilde{B}$, then for each $X \in B_{\mathscr{C}(\mathbb{E})}(X_0, \frac{1}{2})$ the minimization problem $\min(A, X)$ is not well posed. This shows that Theorem 3.3 does not hold, in general, if the space $\mathscr{C}_A(\mathbb{E})$ is replaced by $\mathscr{C}(\mathbb{E})$.

Set $\mathscr{C}^0_A(\mathbb{E}) = \{X \in \mathscr{C}(\mathbb{E}) \mid \lambda_X > 0\}$ and observe that $\mathscr{C}^0_A(\mathbb{E})$ is a Baire space, being completely metrizable by Alexandroff's theorem. Then Theorem 3.3 remains valid with $\mathscr{C}^0_A(\mathbb{E})$, in the place of $\mathscr{C}_A(\mathbb{E})$.

*Remark* 3.5. For $A \in \mathscr{B}(\mathbb{E})$, set $\mathscr{D}_A(\mathbb{E}) = \{X \in \mathscr{C}(\mathbb{E}) \mid X \subset \overline{\mathbb{E} \backslash A}\}$. The space $\mathscr{D}_A(\mathbb{E})$ endowed with the Hausdorff metric is complete and, clearly, $\mathscr{C}_A(\mathbb{E}) \subset \mathscr{D}_A(\mathbb{E})$. Also in the space $\mathscr{D}_A(\mathbb{E})$ Theorem 3.3 is, in general, false. To see that, set $A = \overline{Q \backslash C}$, where $Q = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 3\pi, -1 \leqslant y \leqslant 1\}$ and $C = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 3\pi, -|\sin x| \leqslant y \leqslant |\sin x|\}$, and let $X_0 = \{(x, 0) \in \mathbb{R}^2 \mid \pi/2 \leqslant x \leqslant 5\pi/2\}$. Clearly, $X_0 \in \mathscr{D}_A(\mathbb{R}^2)$. Moreover, if $r > 0$ is sufficiently small, for every $X \in B_{\mathscr{D}_A(\mathbb{R}^2)}(X_0, r)$ the minimization problem $\min(A, X)$ is not well posed.

*Remark* 3.6. Theorem 3.3 remains valid if $A$ is a nonempty closed subset of $\mathbb{E}$, $A \neq \mathbb{E}$. In this case, Theorem 3.3 is a multivalued version of a theorem due to Stečkin [22]. If $\mathbb{E}$ is an arbitrary Banach space, then Theorem 3.3 is, in general, not true. Take, for example, $\mathbb{E} = \mathbb{R}^2$ with the norm $\max\{|x|, |y|\}$, $(x, y) \in \mathbb{R}^2$, and set $A = \tilde{B}$, $X_0 = \{(0, 2)\}$. Then there exists $r > 0$ such that, for every $X \in B_{\mathscr{C}_A(\mathbb{E})}(X_0, r)$, the minimization problem $\min(A, X)$ is not well posed.

*Proof of Lemma* 3.1. Let $X \in \mathscr{C}_A(\mathbb{E})$ and let $r > 0$. We want to show that there exists $Y \in \mathscr{L}_k$ such that $h(Y, X) \leqslant r$. Without loss of generality we suppose $\lambda_X > 0$ and $0 < r < \lambda_X$.

By Proposition 2.4, there exists $0 < \sigma_0 < r$ such that for every $x, y \in \mathbb{E}$ with $\|x - y\| = r$, and for every $0 < \sigma \leqslant \sigma_0$, we have

$$\operatorname{diam} D(x, y; \lambda_X, \sigma) < \varepsilon_k, \tag{3.1}$$

where

$$D(x, y; \lambda_X, \sigma) = \tilde{B}_{\mathbb{E}}(y, \lambda_X - \|y - x\| + \sigma) \backslash B_{\mathbb{E}}(x, \lambda_X).$$

Set

$$\tilde{\sigma} = \min\{\sigma_0, \varepsilon_k\}. \tag{3.2}$$

By Proposition 2.3, there exists $0 < \tau_0 < r/2$ such that for every $u \in \mathbb{E}$ with $d(u, X) \geqslant r/2$, and for every $0 < \tau \leqslant \tau_0$, we have

$$\operatorname{diam} C_{X,u}(\tau) < \frac{\tilde{\sigma}}{2}, \tag{3.3}$$

where $C_{X,u}(\tau)$ is given by (2.3). Set

$$\tilde{\tau} = \min\left\{\tau_0, \frac{\tilde{\sigma}}{2}\right\}. \tag{3.4}$$

Now, pick $\tilde{x} \in X$ and $\tilde{a} \in A$ such that

$$\|\tilde{x} - \tilde{a}\| \leqslant \lambda_X + \frac{\tilde{\tau}}{2}. \tag{3.5}$$

Since $\|\tilde{x} - \tilde{a}\| \geqslant \lambda_X > r$, in the interval with end points $\tilde{x}$ and $\tilde{a}$ there is a point $u$, say, such that

$$\|\tilde{x} - u\| = r. \tag{3.6}$$

Define $Y = \overline{\operatorname{co}}(X \cup \{u\})$. Since $Y \subset \overline{X + rB}$ and $A \cap (X + \lambda_X B) = \phi$, we have $\lambda_Y \geqslant \lambda_X - r > 0$, and so $Y \in \mathscr{C}_A(\mathbb{E})$. Clearly $h(Y, X) \leqslant r$. Thus, to complete the proof, it suffices to show that $Y \in \mathscr{L}_k$.

To this end, we start by proving the following inequalities:

$$\lambda_Y \leqslant \lambda_X + \frac{\tilde{\tau}}{2} - r, \tag{3.7}$$

$$\frac{r}{2} < d(u, X) \leqslant r. \tag{3.8}$$

Indeed, by virtue of (3.5) and (3.6), we have

$$\|u - \tilde{a}\| = \|\tilde{x} - \tilde{a}\| - \|\tilde{x} - u\| \leqslant \lambda_X + \frac{\tilde{\tau}}{2} - r, \tag{3.9}$$

from which (3.7) follows, since $u \in Y$ and $\tilde{a} \in A$. Furthermore, by virtue of (2.1) and (3.9), we have

$$d(u, X) \geqslant \lambda_X - d(u, A) \geqslant \lambda_X - \left(\lambda_X + \frac{\tilde{\tau}}{2} - r\right) = r - \frac{\tilde{\tau}}{2},$$

and thus $d(u, X) > r/2$, for $\tilde{\tau} \leqslant \tau_0 < r/2$. Since the right inequality in (3.8) is trivially satisfied, the proof of (3.8) is complete.

*Claim* 1. We have

$$L_{YA}\left(\frac{\tilde{\tau}}{2}\right) \subset C_{X,u}(\tilde{\tau}). \tag{3.10}$$

Indeed, suppose (3.10) not true, and let $y \in L_{YA}(\tilde{\tau}/2) \backslash C_{X,u}(\tilde{\tau})$ be arbitrary. We have

$$\begin{aligned}
\lambda_X &\leqslant d(y, A) + d(y, X) && \text{(by (2.1))}\\
&\leqslant \lambda_Y + \frac{\tilde{\tau}}{2} + d(y, X) && \text{(as } y \in L_{YA}(\tilde{\tau}/2))\\
&< \lambda_Y + \frac{\tilde{\tau}}{2} + d(u, X) - \tilde{\tau} && \text{(as } y \notin C_{X,u}(\tilde{\tau}))\\
&< \left(\lambda_X + \frac{\tilde{\tau}}{2} - r\right) + \frac{\tilde{\tau}}{2} + r - \tilde{\tau} && \text{(by (3.7) and (3.8))}\\
&= \lambda_X.
\end{aligned}$$

From the contradiction, (3.10) follows and Claim 1 is proved.

*Claim* 2. We have

$$L_{AY}\left(\frac{\tilde{\tau}}{4}\right) \subset D(\tilde{x}, u; \lambda_X, \tilde{\sigma}). \tag{3.11}$$

Indeed, let $a \in L_{AY}(\tilde{\tau}/4)$ be arbitrary. Evidently, $a \in A$ and $d(a, Y) \leqslant \lambda_Y + \tilde{\tau}/4$. Now, pick $y \in Y$ such that $\|a - y\| \leqslant \lambda_Y + \tilde{\tau}/2$. This and (3.7) imply

$$\|a - y\| \leqslant \lambda_X - r + \tilde{\tau}, \tag{3.12}$$

and thus

$$d(y, A) \leqslant \lambda_X - r + \tilde{\tau}. \tag{3.13}$$

By virtue of (2.1), (3.13), and (3.8), we have

$$d(y, X) \geqslant \lambda_X - d(y, A) \geqslant \lambda_X - (\lambda_X - r + \tilde{\tau}) \geqslant d(u, X) - \tilde{\tau},$$

which shows that $y \in C_{X,u}(\tilde{\tau})$. From (3.8) and (3.4), $d(u, X) > r/2$ and $\tilde{\tau} \leqslant \tau_0$. Thus (3.3) gives diam $C_{X,u}(\tilde{\tau}) < \tilde{\sigma}/2$, and so

$$\|y - u\| < \frac{\tilde{\sigma}}{2}. \tag{3.14}$$

Now we have

$$\begin{aligned}\|a-u\| &\leqslant \|a-y\| + \|y-u\| \\ &< (\lambda_X - r + \tilde{\tau}) + \frac{\tilde{\sigma}}{2} \qquad \text{(by (3.12), (3.14))} \\ &\leqslant \lambda_X - \|\tilde{x}-u\| + \tilde{\sigma} \qquad \text{(by (3.6), (3.4))},\end{aligned}$$

which shows that $a \in \tilde{B}_{\mathbb{E}}(u, \lambda_X - \|\tilde{x}-u\| + \tilde{\sigma})$. Clearly $\|a-\tilde{x}\| \geqslant \lambda_X$, that is, $a \notin B_{\mathbb{E}}(\tilde{x}, \lambda_X)$. Hence $a \in D(\tilde{x}, u; \lambda_X, \tilde{\sigma})$. As $a \in L_{AY}(\tilde{\tau}/4)$ is arbitrary, (3.11) is proved, completing the proof of Claim 2.

As diam $C_{X,u}(\tilde{\tau}) < \tilde{\sigma}/2$ and, by (3.2), $\tilde{\sigma} \leqslant \varepsilon_k$, from Claim 1 we have

$$\operatorname{diam} L_{YA}\left(\frac{\tilde{\tau}}{2}\right) < \varepsilon_k. \tag{3.15}$$

Furthermore, from (3.6) and (3.2), $\|\tilde{x}-u\| = r$ and $\tilde{\sigma} \leqslant \sigma_0$. Thus (3.1) gives diam $D(\tilde{x}, u; \lambda_X, \tilde{\sigma}) < \varepsilon_k$. Hence, by Claim 2, we have

$$\operatorname{diam} L_{AY}\left(\frac{\tilde{\tau}}{4}\right) < \varepsilon_k. \tag{3.16}$$

From (3.15) and (3.16), it follows that $Y \in \mathscr{L}_k$, which completes the proof of Lemma 3.1.

*Proof of Lemma* 3.2. Indeed, let $X \in \mathscr{L}_k$ be arbitrary. Let $\eta > 0$ be such that

$$\theta + 2\eta < \varepsilon_k, \qquad \text{where} \quad \theta = \min\{\inf_{\sigma>0} \operatorname{diam} L_{XA}(\sigma), \inf_{\sigma>0} \operatorname{diam} L_{AX}(\sigma)\}. \tag{3.17}$$

Furthermore, let $\sigma_1 > 0$ be such that

$$\operatorname{diam} L_{XA}(\sigma_1) < \theta + \eta, \qquad \operatorname{diam} L_{AX}(\sigma_1) < \theta + \eta. \tag{3.18}$$

Fix $\sigma_2$, $0 < \sigma_2 < \sigma_1$, and set

$$\delta = \min\left\{\frac{\sigma_1 - \sigma_2}{2}, \frac{\eta}{2}\right\}. \tag{3.19}$$

We claim that $B_{\mathscr{C}_A(\mathbb{E})}(X, \delta) \subset \mathscr{L}_k$. To prove that, let $Y \in B_{\mathscr{C}_A(\mathbb{E})}(X, \delta)$ be arbitrary. Let $y \in L_{YA}(\sigma_2)$ be arbitrary. As $h(Y, X) < \delta$, there exists an $x \in X$ such that $\|y - x\| < \delta$. We have

$$\begin{aligned} d(x, A) &< d(y, A) + \delta \\ &\leqslant \lambda_Y + \sigma_2 + \delta && (\text{as } y \in L_{YA}(\sigma_2)) \\ &< (\lambda_X + \delta) + \sigma_2 + \delta && (\text{as } h(Y, X) < \delta) \\ &\leqslant \lambda_X + \sigma_1 && (\text{by } (3.19)), \end{aligned}$$

and so $x \in L_{XA}(\sigma_1)$. Hence $y = x + (y - x) \in L_{XA}(\sigma_1) + \delta B$, from which, since $y$ is arbitrary in $L_{YA}(\sigma_2)$, we have $L_{YA}(\sigma_2) \subset L_{XA}(\sigma_1) + \delta B$. From this, by virtue of (3.18), (3.19), and (3.17), we have

$$\operatorname{diam} L_{YA}(\sigma_2) \leqslant \operatorname{diam} L_{XA}(\sigma_1) + 2\delta < \theta + 2\eta < \varepsilon_k. \tag{3.20}$$

Now, let $a \in L_{AY}(\sigma_2)$ be arbitrary. We have $d(a, X) \leqslant d(a, Y) + h(Y, X)$. From this, it follows that

$$\begin{aligned} d(a, X) &< d(a, Y) + \delta && (\text{as } h(Y, X) < \delta) \\ &\leqslant (\lambda_Y + \sigma_2) + \delta && (\text{as } a \in L_{AY}(\sigma_2)) \\ &< (\lambda_X + \delta) + \sigma_2 + \delta && (\text{as } h(Y, X) < \delta) \\ &\leqslant \lambda_X + \sigma_1 && (\text{by } (3.19)), \end{aligned}$$

which shows that $a \in L_{AX}(\sigma_1)$. As $a \in L_{AY}(\sigma_2)$ is arbitrary, we have $L_{AY}(\sigma_2) \subset L_{AX}(\sigma_1)$. From this, by virtue of (3.18) and (3.17), we have

$$\operatorname{diam} L_{AY}(\sigma_2) \leqslant \operatorname{diam} L_{AX}(\sigma_1) < \theta + \eta < \varepsilon_k. \tag{3.21}$$

From (3.20) and (3.21) it follows that $Y \in \mathscr{L}_k$. As $Y \in B_{\mathscr{C}_A(\mathbb{E})}(X, \delta)$ is arbitrary, the proof of Lemma 3.2 is complete.

## 4. Maximization Problems

Also in this section $\mathbb{E}$ denotes a uniformly convex Banach space. Let $A$ be a fixed nonempty closed bounded subset of $\mathbb{E}$. We put, for short, $\mu_X = \mu_{AX}$, $X \in \mathscr{B}(\mathbb{E})$.

For each $k \in \mathbb{N}$, set $\varepsilon_k = 1/k$, and define

$$\mathscr{M}_k = \{X \in \mathscr{C}(\mathbb{E}) \mid \inf_{\sigma > 0} \operatorname{diam} M_{XA}(\sigma) < \varepsilon_k \quad \text{and} \quad \inf_{\sigma > 0} \operatorname{diam} M_{AX}(\sigma) < \varepsilon_k\}.$$

To prove the main result of this section, Theorem 4.3, we state two lemmas whose proofs will be given later.

LEMMA 4.1. *$\mathcal{M}_k$ is dense in $\mathscr{C}(\mathbb{E})$.*

LEMMA 4.2. *$\mathcal{M}_k$ is open in $\mathscr{C}(\mathbb{E})$.*

THEOREM 4.3. *Let $\mathbb{E}$ be a uniformly convex Banach space. Let $A \in \mathscr{B}(\mathbb{E})$. Then the set*

$$\mathscr{V} = \{X \in \mathscr{C}(\mathbb{E}) \mid \max(A, X) \text{ is well posed}\}$$

*is a dense $G_\delta$-subset of $\mathscr{C}(\mathbb{E})$.*

*Proof.* By Lemmas 4.1 and 4.2, the set

$$\mathcal{M}_0 = \bigcap_{k \in \mathbb{N}} \mathcal{M}_k$$

is a dense $G_\delta$-subset of $\mathscr{C}(\mathbb{E})$. Moreover, by Proposition 2.2, we have $\mathscr{V} = \mathcal{M}_0$. Hence $\mathscr{V}$ is a dense $G_\delta$-subset of $\mathscr{C}(\mathbb{E})$, completing the proof.

*Remark* 4.4. Theorem 4.3 is a multivalued version of results due to Asplund [1] and Edelstein [11]. Note also that with the notation of the example given in Remark 3.6, there exists $r > 0$ such that, for every $X \in B_{\mathscr{C}(\mathbb{E})}(X_0, r)$, the maximization problem $\max(A, X)$ is not well posed. This shows that, if $\mathbb{E}$ is an arbitrary Banach space, then Theorem 4.3 is, in general, not true.

*Proof of Lemma* 4.1. Let $X \in \mathscr{C}(\mathbb{E})$ and let $r > 0$. We want to show that there exists $Y \in \mathcal{M}_k$ such that $h(Y, X) \leqslant r$. The case $\mu_X = 0$ is trivial. Thus, without loss of generality, we suppose $\mu_X > 0$ and take $r$ such that $0 < r < \mu_X$.

By Proposition 2.4, there exists $0 < \sigma_0 < r$ such that for every $x, y \in \mathbb{E}$, with $\|y - x\| = r$, and for every $0 < \sigma \leqslant \sigma_0$, we have

$$\operatorname{diam} D(x, y; \mu_X + \|y - x\| - \sigma, \sigma) < \varepsilon_k, \tag{4.1}$$

where

$$D(x, y; \mu_X + \|y - x\| - \sigma, \sigma) = \tilde{B}_{\mathbb{E}}(y, \mu_X) \backslash B_{\mathbb{E}}(x, \mu_X + \|y - x\| - \sigma).$$

Set

$$\tilde{\sigma} = \min\{\sigma_0, \varepsilon_k\}. \tag{4.2}$$

By Proposition 2.3, there exists $0 < \tau_0 < r/2$ such that for every $u \in \mathbb{E}$, with $d(u, X) \geqslant r/2$, and for every $0 < \tau \leqslant \tau_0$, we have

$$\operatorname{diam} C_{X,u}(\tau) < \frac{\tilde{\sigma}}{2}, \tag{4.3}$$

where $C_{X,u}(\tau)$ is given by (2.3). Set

$$\tilde{\tau} = \min\left\{\tau_0, \frac{\tilde{\sigma}}{2}\right\}. \tag{4.4}$$

Now, pick $\tilde{x} \in X$ and $\tilde{a} \in A$ such that $\|\tilde{x} - \tilde{a}\| \geqslant \mu_X - \tilde{\tau}/4$, and observe that $\tilde{x} \neq \tilde{a}$, for $\mu_X > r > \sigma_0 \geqslant \tilde{\sigma} > \tilde{\tau}$. Set

$$u = \tilde{x} + r\,\frac{\tilde{x} - \tilde{a}}{\|\tilde{x} - \tilde{a}\|}, \qquad Y = \overline{\mathrm{co}}(X \cup \{u\}).$$

Clearly $Y \in \mathscr{C}(\mathbb{E})$, and $h(Y, X) \leqslant r$. Thus, to complete the proof it suffices to show that $Y \in \mathscr{M}_k$.

To this end, we start by proving the following inequalities:

$$\mu_Y \geqslant \mu_X + r - \frac{\tilde{\tau}}{4}, \tag{4.5}$$

$$d(u, X) \geqslant r - \frac{\tilde{\tau}}{4}. \tag{4.6}$$

Indeed, $\|u - \tilde{a}\| = \|u - \tilde{x}\| + \|\tilde{x} - \tilde{a}\| \geqslant r + (\mu_X - \tilde{\tau}/4)$, from which (4.5) follows, for $u \in Y$ and $\tilde{a} \in A$. Furthermore, from (2.2) we have

$$d(u, X) \geqslant e(u, A) - \mu_X \geqslant \left(\mu_X + r - \frac{\tilde{\tau}}{4}\right) - \mu_X = r - \frac{\tilde{\tau}}{4},$$

for $e(u, A) \geqslant \mu_X + r - \tilde{\tau}/4$, and so also (4.6) is proved.

*Claim* 1. We have

$$M_{YA}\left(\frac{\tilde{\tau}}{2}\right) \subset C_{X,u}(\tilde{\tau}). \tag{4.7}$$

Indeed, suppose (4.7) false, and let $y \in M_{YA}(\tilde{\tau}/2) \setminus C_{X,u}(\tilde{\tau})$ be arbitrary. From the definition of $M_{YA}(\tilde{\tau}/2)$ and from (4.5), we have

$$e(y, A) \geqslant \mu_Y - \frac{\tilde{\tau}}{2} \geqslant \mu_X + r - \frac{3}{4}\tilde{\tau}. \tag{4.8}$$

On the other hand, we have

$$\begin{aligned} e(y, A) &\leqslant \mu_X + d(y, Y) && \text{(by (2.2))} \\ &< \mu_X + (d(u, X) - \tilde{\tau}) && (\text{as } y \in Y \setminus C_{X,u}(\tilde{\tau})) \\ &\leqslant \mu_X + r - \tilde{\tau} && (\text{as } d(u, X) \leqslant r). \end{aligned}$$

Since the latter inequality contradicts (4.8), Claim 1 is true.

*Claim* 2. We have

$$M_{AY}\left(\frac{\tilde{\tau}}{4}\right) \subset D(u, \tilde{x}; \mu_X + \|\tilde{x}-u\| - \tilde{\sigma}, \tilde{\sigma}). \tag{4.9}$$

Indeed, let $a \in M_{AY}(\tilde{\tau}/4)$ be arbitrary. As $e(a, Y) \geq \mu_Y - \tilde{\tau}/4$, there exists $y \in Y$ such that

$$\|y-a\| \geq \mu_Y - \frac{\tilde{\tau}}{2}. \tag{4.10}$$

By (2.2) we have $d(y, X) \geq e(y, A) - \mu_X$, from which, by using (4.10) and (4.5), we get

$$d(y, X) \geq \left(\mu_Y - \frac{\tilde{\tau}}{2}\right) - \mu_X \geq \left(\mu_X + r - \frac{\tilde{\tau}}{4}\right) - \frac{\tilde{\tau}}{2} - \mu_X > r - \tilde{\tau}.$$

From this, since $r \geq d(u, X)$, we have $d(y, X) > d(u, X) - \tilde{\tau}$, and so $y \in C_{X,u}(\tilde{\tau})$. From (4.4) and (4.6) we have $\tilde{\tau} \leq \tau_0$ and $d(u, X) > r/2$. But, by (4.3), diam $C_{X,u}(\tilde{\tau}) < \tilde{\sigma}/2$, which implies

$$\|y-u\| < \frac{\tilde{\sigma}}{2}. \tag{4.11}$$

Now, we have

$$\begin{aligned} \|a-u\| &\geq \|a-y\| - \|y-u\| \\ &> \left(\mu_Y - \frac{\tilde{\tau}}{2}\right) - \frac{\tilde{\sigma}}{2} && \text{(by (4.10) and (4.11))} \\ &\geq \left(\mu_X + r - \frac{\tilde{\tau}}{4}\right) - \frac{\tilde{\tau}}{2} - \frac{\tilde{\sigma}}{2} && \text{(by (4.5))} \\ &> \mu_X + r - \tilde{\sigma} && \text{(by (4.4))}. \end{aligned}$$

Hence $a \notin B_{\mathbb{E}}(u, \mu_X + \|\tilde{x}-u\| - \tilde{\sigma})$, for $\|\tilde{x}-u\| = r$. Clearly, $a \in \tilde{B}_{\mathbb{E}}(\tilde{x}, \mu_X)$. Hence $a \in D(u, \tilde{x}; \mu_X + \|\tilde{x}-u\| - \tilde{\sigma}, \tilde{\sigma})$. As $a \in M_{AY}(\tilde{\tau}/4)$ is arbitrary, (4.9) is proved, completing the proof of Claim 2.

As diam $C_{X,u}(\tilde{\tau}) < \tilde{\sigma}/2$ and, by (4.2), $\tilde{\sigma} \leq \varepsilon_k$, Claim 1 gives

$$\operatorname{diam} M_{YA}\left(\frac{\tilde{\tau}}{2}\right) < \varepsilon_k. \tag{4.12}$$

Furthermore, from (4.1) we have diam $D(u, \tilde{x}; \mu_X + \|\tilde{x} - u\| - \tilde{\sigma}, \tilde{\sigma}) < \varepsilon_k$, since $\|\tilde{x} - u\| = r$ and, by (4.2), $\tilde{\sigma} \leqslant \sigma_0$. Hence, by Claim 2,

$$\operatorname{diam} M_{AY}\left(\frac{\tilde{\tau}}{4}\right) < \varepsilon_k.$$

From (4.12) and the latter inequality it follows that $Y \in \mathcal{M}_k$, which completes the proof of Lemma 4.1.

*Proof of Lemma* 4.2. This is similar to the proof of Lemma 3.2, and so it is omitted.

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